Models of Set Theory II - Winter 2013

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Problem 25 (8 Points)**.** Let end_T denote the set of end nodes of a tree T. A *Borel code* for a subset of ${}^\omega\omega$ is a pair (T, f, g), where T is a subtree of ${}^{<\omega}\omega$ with no infinite branches and $f\colon T \setminus end_T \to 3$, $g\colon end_T \to {}^{<\omega}\omega$ are functions. If (T, f, g) is a Borel code and $t \in T$, the set $\mathcal{B}_{(T,f,g,t)}$ is defined as

(i) $U_{g(t)} := \{x \in {}^\omega\omega \mid g(t) \subseteq x\}$ if $t \in end_T$,
(ii) ${}^\omega\omega \setminus \mathcal{B}_{(T,f,g,t^\frown i)}$ if $f(t) = 0$ and i is least with $t^\frown i \in T$,
(iii) $\bigcup_{t^\frown i\in T} \mathcal{B}_{(T,f,g,t^\frown i)}$ if $f(t) = 1$,
(iv) $\bigcap_{t^\frown i\in T} \mathcal{B}_{(T,f,g,t^\frown i)}$ if $f(t) = 2$,

and $\mathcal{B}_{(T,f,g)} := \mathcal{B}_{(T,f,g,\emptyset)}$ denote the *Borel set coded by* (T, f, g). Suppose that $M \subseteq N$ are transitive models of ZFC.

(a) Suppose that T is a subtree of ${}^{<\omega}\omega$ in M. Show that T has no infinite branches in M if and only if T has no infinite branches in N.
(b) Suppose that (T, f, g) is a Borel code in M. Show that (T, f, g) is a Borel code in N and that

$$M \models \text{"}\mathcal{B}_{(T,f,g)} \text{ is meager"} \iff N \models \text{"}\mathcal{B}_{(T,f,g)} \text{ is meager"}.$$

(*You should use without proof the fact that for all Borel codes* (T, f, g), (T', f', g') *in* M*, the statement* "$\mathcal{B}_{(T,f,g)} \subseteq \mathcal{B}_{(T',f',g')}$" *is absolute between* M *and* N.)

Problem 26 (6 Points)**.** Suppose that (X, d) is Polish space, i.e. X is separable and d is a complete metric on X. Suppose that $(U_s)_{s\in{}^{<\omega}\omega}$ is a family of subsets of X with the following properties.

(i) $U_{s^\frown i} \cap U_{s^\frown j} = \emptyset$ for $i \neq j$,
(ii) $U_{s^\frown i} \subseteq U_s$,
(iii) U_s is closed and open, and
(iv) $\lim_{n\to\infty} \operatorname{diam}(U_{x\restriction n}) = 0$ for all $x \in {}^\omega\omega$.

Let $D = \{x \in {}^\omega\omega \mid \bigcap_{n\in\omega} U_{x\restriction n} \neq \emptyset\}$ and $f\colon D \to {}^\omega\omega$, where $f(x)$ is the unique element of $\bigcap_{n\in\omega} U_{x\restriction n}$. Prove that D is closed and that f is a homeomorphism onto its image.

Problem 27 (4 Points)**.** (a) Suppose that C is a countable dense set of the real line $\mathbb{R}$. Show that $\mathbb{R} \setminus C$ is homeomorphic to the Baire space ${}^\omega\omega$.
(b) Show that the cardinal characteristics associated to the meager ideals on ${}^\omega\omega$ and $\mathbb{R}$ are equal.

Problem 28 (4 Points)**.** Let $\mathfrak{u}$ denote the least cardinality of a family of subsets of ω which generates a nonprincipal ultrafilter on ω. Show that $\mathfrak{b} \leq \mathfrak{u}$.
(*Hint: For* $X \in [\omega]^\omega$*, let* g_X *denote the increasing enumeration of* X*. For an increasing function* $f\colon \omega \to \omega$*, let* S_f *denote the union of the intervals* $[f^{2n}(0), f^{2n+1}(0))$ *for* $n \in \omega$*. Show that if an increasing function* $f\colon \omega \to \omega$ *eventually dominates* g_X*, then both* $S_f \cap X$ *and* $S_f \setminus X$ *are infinite.*)

Please hand in your solutions on Monday, December 09 before the lecture.